On the mean square of the error term for the two-dimensional divisor problems(II)

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Abstract. Let $\Delta(a,b;x)$ denote the error term of the general two-dimensional divisor problem. In this paper we shall study the relation between the discrete mean value $\sum_{n\le T}\Delta^2(a,b;n)$ and the continuous mean value $\int_1^T\Delta^2(a,b;x)dx$.

1 Introduction and state of results

Suppose $1\le a\le b$ are two fixed integers. Without loss of generality, we suppose $(a,b)=1$. Define

$$d(a,b;n):=\sum_{n=h^ar^b}1,\qquad D(a,b;x):=\sum_{n\le x}d(a,b;n).$$

The two-dimensional divisor problems is to study the error term

$$\Delta(a,b;x):=D(a,b;x)-\begin{cases}\left(\zeta(b/a)x^{1/a}+\zeta(a/b)x^{1/b}\right), & \text{if } 1\le a<b,\\ \left(x\log x+(2\gamma-1)x\right), & \text{if } a=b=1.\end{cases}\tag{1.1}$$

When $a=b=1$, $\Delta(1,1;x)$ is the error term of the well-known Dirichlet divisor problem. Dirichlet first proved that $\Delta(1,1;x)=O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(1,1;x)\ll x^{131/416}\log^{26947/8320}x,\tag{1.2}$$

which can be found in Huxley[11]. It is conjectured that

$$\Delta(1,1;x)=O(x^{1/4+\varepsilon}),\tag{1.3}$$

2000 Mathematics Subject Classification: 11N37.

Key Words: two-dimensional divisor problems , error term, mean square, asymptotic formula .

This work is supported by National Natural Science Foundation of China(Grant No. 10771127) .

which is supported by the classical mean-square result

$$\int_1^T \Delta^2(1,1;x)dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + O(T^{5/4+\varepsilon}) \tag{1.4}$$

proved in [3]. The estimate $O(T^{5/4+\varepsilon})$ was improved to $O(T\log^5 T)$ in [27] and $O(T\log^4 T)$ in [23]. The mean square of the error term in (1.4) was studied in [20] and [29]. The higher-power moments of $\Delta(1,1;x)$ were studied in[9, 16, 28, 33, 34].

When $a \neq b$, Richert [25] proved that

$$\Delta(a,b;x) \ll \begin{cases} x^{\frac{2}{3(a+b)}}, & \text{if } b \leq 2a \\ x^{\frac{2}{5a+2b}}, & \text{if } b \geq 2a, \end{cases} \tag{1.5}$$

Better upper estimates can be found in [17, 18, 24, 26]. Hafner [7] showed that

$$\Delta(a,b;x) = \Omega_+\left(x^{\frac{1}{2(a+b)}}(\log x)^{\frac{b}{2(a+b)}}\log\log x\right), \tag{1.6}$$

and

$$\Delta(a,b;x) = \Omega_-\left(x^{\frac{1}{2(a+b)}}e^{U(x)}\right), \tag{1.7}$$

where

$$U(x) = B\,(\log\log x)^{\frac{b}{2(a+b)}}\,(\log\log\log x)^{\frac{b}{2(a+b)}-1}, \tag{1.8}$$

for some constant $B > 0$.

For $1 \leq a < b$ it is conjectured that the estimate

$$\Delta(a,b;x) = O\left(x^{\frac{1}{2(a+b)}+\varepsilon}\right) \tag{1.9}$$

holds for $x \geq 2$, which is supported partially by results of Ivić[13]. Ivić showed that

$$\int_1^T \Delta^2(a,b;x)dx \begin{cases} \ll T^{1+\frac{1}{a+b}}\log^2 T, \\ = \Omega(T^{1+\frac{1}{a+b}}). \end{cases} \tag{1.10}$$

The Ω result in (1.10) was improved by the first-named author[1] to

$$\int_1^T \Delta^2(a,b;x)dx \gg T^{1+\frac{1}{a+b}}. \tag{1.11}$$

Ivić[13] conjectured that the asymptotic formula

$$\int_1^T \Delta^2(a,b;x)dx = c_{a,b}(1+o(1))T^{1+\frac{1}{a+b}} \tag{1.12}$$

should hold for some constant $c_{a,b} > 0$. This conjecture was solved completely in [32], where we proved that if $1 \le a < b$ and $(a,b) = 1$, then for $T \ge 2$

$$\int_1^T \Delta^2(a,b;x)dx = c_{a,b}T^{\frac{1+a+b}{a+b}} + O(T^{\frac{1+a+b}{a+b} - \frac{a}{2b(a+b)(a+b-1)}} \log^{7/2} T), \tag{1.13}$$

where

$$c_{a,b} := \frac{a^{b/(a+b)}b^{a/(a+b)}}{2(a+b+1)\pi^2} \sum_{n=1}^{\infty} g_{a,b}^2(n)$$

and

$$g_{a,b}(n) := \sum_{n=h^a r^b} h^{-\frac{a+2b}{2a+2b}} r^{-\frac{b+2a}{2a+2b}}.$$

The aim of this paper is to study the relation between discrete mean and continuous mean of $\Delta(a,b;x)$. This kind of problem is very important and interesting in number theory.

Voronoï[31] essentially showed that for $x \ge 1$, the asymptotic formula

$$\sum_{n \le x} \Delta(1,1;n) = \left(\frac{1}{2} - \psi(x)\right) \Delta(1,1;x) + \int_1^x \Delta(1,1;t)dt \tag{1.14}$$

$$+\frac{1}{2}(\log x + 2\gamma - 1)\, x + O(\log x)$$

holds. For the mean square case, Hardy[8] proved that

$$\sum_{n \le x} \Delta^2(1,1;n) = \int_1^x \Delta^2(1,1;t)dt + O(x^{1+\varepsilon}),$$

which was improved by Furuya [5] substantially to

$$\begin{aligned} \sum_{n \le x} \Delta^2(1,1;n) &= \int_1^x \Delta^2(1,1;t)dt + \frac{1}{6}x \log^2 x \\ &\quad + \frac{8\gamma - 1}{12} x \log x + \frac{8\gamma^2 - 2\gamma + 1}{12} x + \begin{cases} O(x^{\frac{3}{4}} \log x), \\ \Omega_{\pm}(x^{\frac{3}{4}} \log x), \end{cases} \end{aligned}$$

For $1 \le a < b$, following Voronoï[31] we can get

$$\sum_{n \le x} \Delta(a,b;n) = \left(\frac{1}{2} - \psi(x)\right) \Delta(a,b;x) + \int_1^x \Delta(a,b;t)dt \tag{1.15}$$

$$+\frac{1}{2}\left(\zeta(\frac{b}{a})x^{\frac{1}{a}}+\zeta(\frac{a}{b})x^{\frac{1}{b}}\right)+O(1).$$

From (1.14) and (1.15) we may write

$$\begin{aligned}\sum_{n\le x}\Delta(a,b;n) \quad &= \frac{1}{2}D(a,b;x)-\psi(x)\Delta(a,b;x)+\int_1^x\Delta(a,b;t)dt\\ &+\begin{cases} O(\log x), & \text{if } a=b=1,\\ O(1), & \text{if } 1\le a<b,\end{cases}\end{aligned}$$

which combining (1.5) and Lemma 2.2 implies that

$$\sum_{n\le x}\Delta(1,b;n)=\left(\frac{1}{4}+\frac{1}{2}\zeta(b)\right)x+\begin{cases} O\left(x^{1-\frac{1}{2(1+b)}}\right)\\ \Omega_{\pm}\left(x^{1-\frac{1}{2(1+b)}}\right)\end{cases} \tag{1.16}$$

for $b>1$, and implies that

$$\sum_{n\le x}\Delta(a,b;n)=\frac{1}{4}x+\begin{cases} O\left(x^{1-\frac{1}{2(a+b)}}\right),\\ \Omega_{\pm}\left(x^{1-\frac{1}{2(a+b)}}\right),\end{cases} \tag{1.17}$$

for $2\le a<b$. We omit the proofs of (1.15)-(1.17).

We shall study the mean square case for $1\le a<b$. Our main result is the following

Theorem 1. Let $1\le a<b, (a,b)=1$, and $x\ge 2$, then we have

$$\sum_{n\le x}\Delta^2(a,b;n)= \quad \left(\frac{1}{2}-\psi(x)\right)\Delta^2(a,b;x)+\int_1^x\Delta^2(a,b;t)dt \tag{1.18}$$

$$+\frac{1}{4}\left(\zeta(\frac{b}{a})x^{\frac{1}{a}}+\zeta(\frac{a}{b})x^{\frac{1}{b}}\right)+\left(\frac{1}{a}\zeta(\frac{b}{a})x^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})x^{\frac{1}{b}-1}\right)G_{(a,b)}(x)$$

$$+\begin{cases} \frac{1}{6}\zeta(b)\left(\zeta(b)x+2\zeta(\frac{1}{b})x^{\frac{1}{b}}\right)+O(x^{1-\frac{3}{2(1+b)}}), & \text{if } 1=a<b,\\ O(x^{\frac{1}{a}-\frac{3}{2(a+b)}}), & \text{if } 2\le a<b,\end{cases}$$

where

$$G_{(a,b)}(x):=c_0x^{1-\frac{1}{2(a+b)}}\sum_{n=1}^{\infty}\frac{d^*(a,b;n)}{n^{1+\frac{1}{2(a+b)}}}\cos\left(2(a+b)\pi(a^{-a}b^{-b}nx)^{\frac{1}{a+b}}+\theta_0\right), \tag{1.19}$$

$$d^*(a,b;n) = \sum_{m^a n^b = n} m^{a-1} n^{b-1}, \tag{1.20}$$

$$c_0 = \frac{(a^a b^b)^{1+\frac{1}{2(a+b)}}}{2\pi^2\sqrt{ab(a+b)}}, \quad \theta_0 = -\frac{3\pi}{4}. \tag{1.21}$$

By Theorem 1 and (1.5), it is easy to see that

Corollary 1. Suppose $1 \le a < b, (a,b) = 1$ and $x \ge 2$, then we have

$$\sum_{n\le x} \Delta^2(a,b;n) = \int_1^x \Delta^2(a,b;t)dt + R^*(a,b;x) \tag{1.22}$$

$$+\begin{cases} \frac{1}{12}\zeta(b)(3+2\zeta(b))x, & \text{if } 1 = a < b, \\ \frac{1}{4}\zeta(\frac{b}{a})x^{\frac{1}{a}} + \frac{1}{4}\zeta(\frac{a}{b})x^{\frac{1}{b}}, & \text{if } 2 \le a < b, \end{cases}$$

where $R^*(a,b;x) = O\left(x^{\frac{1}{a}-\frac{1}{2(a+b)}}\right)$ and $R^*(a,b;x) = \Omega_\pm\left(x^{\frac{1}{a}-\frac{1}{2(a+b)}}\right)$.

Remark. Generally speaking, the term $\frac{1}{4}\zeta(\frac{a}{b})x^{\frac{1}{b}}$ in Corollary 1 can not be removed since $\frac{1}{b} > \frac{1}{a} - \frac{1}{2(a+b)}$ for $b < \frac{1+\sqrt{17}}{4}a$.

From Corollary 1 and (1.13) we get

Corollary 2. Suppose $1 \le a < b, (a,b) = 1$ and $x \ge 2$, then

$$\sum_{n\le x} \Delta^2(a,b;n) = c_{a,b}x^{\frac{1+a+b}{a+b}} + O(x^{\frac{1+a+b}{a+b}-\frac{a}{2b(a+b)(a+b-1)}}\log^{7/2} x).$$

We have also the following Theorem 2, which slightly improves Furuya's result.

Theorem 2. For $x \ge 2$, we have

$$\sum_{n\le x} \Delta^2(1,1;n) = \left(\frac{1}{2} - \psi(x)\right)\Delta^2(1,1;x) + \int_1^x \Delta^2(1,1;t)dt \tag{1.23}$$

$$+\frac{1}{6}x\log^2 x + \frac{8\gamma-1}{12}x\log x + \frac{8\gamma^2-2\gamma+1}{12}x + (\log x + 2\gamma)G_{(1,1)}(x) + O(\sqrt{x}\log x),$$

where

$$G_{(1,1)}(x) := \frac{1}{2\sqrt{2}\pi^2}x^{\frac{3}{4}}\sum_{n=1}^\infty \frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right).$$

Our proof is based on the method of Furuya [5]. We need a sharper asymptotic formula for the error term $\Delta(a,b;x)$, and then evaluate a kind of integrals involving the ψ-function.

Notations. For a real number u, $[u]$ denotes the greatest integer not exceeding u, $\psi(u) = u-[u]-1/2$. Let (m,n) denote the greatest common divisor of natural numbers m and n. $n \sim N$ means $N < n \le 2N$. ε always denotes a sufficiently small positive constant. In this paper, the constants implied by O depend olny on a, b or ε when it occures. $f(x) = O(g(x))$ or $f(x) \ll g(x)$ means that $|f(x)| \le Cg(x)$ for $x \ge x_0$ and some absolute constant $C > 0$. $f(x) = \Omega_{\pm}(g(x))$ means that both $f(x) = \Omega_{+}(g(x))$ and $f(x) = \Omega_{-}(g(x))$ holds.

2 Some preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. Let $f(n)$ be an arithmetic function, and $E(x)$ be the error term defined by

$$E(x) := \sum_{n\le x} f(n) - g(x). \tag{2.1}$$

Suppose $g(x)$ is continuously differentiable. For any fixed positive integer k, we have

$$\sum_{n\le x} E^k(n) = \left(\frac{1}{2}-\psi(x)\right)E^k(x) + \int_1^x E^k(u)du + k\int_1^x \left(\frac{1}{2}-\psi(u)\right)g'(u)E^{k-1}(u)du. \tag{2.2}$$

Proof. This is Lemma 1 of Furuya [5].

Lemma 2.2. (Voronoï type formula) Let $x \ge 1$, and $d^*(a,b;n)$ is defined by (1.20), then for every fixed positive integer q, the following asymptotic formula holds

$$\int_0^x \Delta(a,b;t)dt = \frac{x}{4} + \zeta(-a)\zeta(-b) + O(x^{1-\frac{1}{2(a+b)}-\frac{q}{a+b}}) + \sum_{m=0}^{q-1} c_m \left(\sum_{n=1}^{\infty} \frac{d^*(a,b;n)}{n^{1+\frac{1}{2(a+b)}+\frac{m}{a+b}}} \cos\left(2(a+b)\pi(a^{-a}b^{-b}nx)^{\frac{1}{a+b}}+\theta_m\right)\right) x^{1-\frac{1}{2(a+b)}-\frac{m}{a+b}}, \tag{2.3}$$

where c_m and θ_m are real numbers, c_0 and θ_0 are defined in Theorem 1.

Proof. This is Theorem 3 of the first-named author [1]. We note that when $a = b = 1$, Lemma 2.2 was already proved by Tong [27].

Now for $x \ge 1$, we define the $\psi_j(x)$ by the following recurrence relation

$$\psi_j(x) := \int_1^x \psi_{j-1}(t)dt, (j = 1, 2, \cdots), \tag{2.4}$$

for convenience, we use the notation $\psi_0(x) = \psi(x)$ here.

Lemma 2.3. For $x \geq 1$, we have

$$\int_1^x \psi^{2k-1}(t)dt = \frac{1}{2k}\left(\psi^{2k}(x) - \frac{1}{2^{2k}}\right), (k = 1, 2, \cdots) \tag{2.5}$$

and

$$\int_1^x \psi^{2k}(t)dt = \frac{1}{2k+1}\left(\frac{x-1}{2^{2k}} - \frac{1}{2^{2k}}\psi(x) + \psi^{2k+1}(x)\right), (k = 1, 2, \cdots). \tag{2.6}$$

Proof. We prove (2.6) only, the proof of (2.5) is similar. By simple calculations, we get

$$\begin{aligned}\int_1^x \psi^{2k}(t)dt \quad &= \sum_{n=0}^{[x]-1} \int_n^{n+1} \psi^{2k}(t)dt + \int_{[x]}^x \psi^{2k}(t)dt \\ &= ([x]-1)\int_1^2 \psi^{2k}(t)dt + \int_{[x]}^x (t - [x] - \frac{1}{2})^{2k} dt \\ &= 2([x]-1)\int_0^{\frac{1}{2}} u^{2k} du + \int_{-\frac{1}{2}}^{\psi(x)} u^{2k} du \\ &= ([x]-1)\frac{1}{2^{2k}(2k+1)} + \frac{1}{(2k+1)}\left(\psi^{2k+1}(x) + \frac{1}{2^{2k+1}}\right),\end{aligned}$$

and whence (2.6) follows.

From Lemma 2.3, we easily get that for $x \geq 1$

$$\psi_1(x) = \frac{1}{2}(\psi^2(x) - \frac{1}{4}), \tag{2.7}$$

and

$$\psi_2(x) = -\frac{1}{12}x + \frac{1}{6}\psi^3(x) - \frac{1}{24}\psi(x) + \frac{1}{12}. \tag{2.8}$$

Lemma 2.4. Let $x \geq 1$ and define $W_\alpha(x) := \int_1^x t^\alpha \psi(t)dt$. If $\alpha \neq -1, -2$, we have

$$W_\alpha(x) = -\frac{x^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+1}\sum_{n \leq x} n^{\alpha+1} + \tag{2.9}$$

$$\frac{\psi(x)}{\alpha+1}x^{\alpha+1} - \frac{\alpha}{2(\alpha+1)(\alpha+2)},$$

$$W_{-2}(x) = \log x - \psi(x)x^{-1} - \sum_{n\le x} n^{-1} + \frac{1}{2} \tag{2.10}$$

$$= \frac{1}{2} - \gamma + \left(\psi_1(x) + \frac{1}{12}\right) x^{-2} + O(x^{-3}), \tag{2.11}$$

and

$$W_{-1}(x) = 2\int_1^\infty \frac{\psi_2(t)}{t^3}dt + \left(\psi_1(x) + \frac{1}{12}\right) x^{-1} + O(x^{-2}). \tag{2.12}$$

Proof. First, we suppose $x \ge 2$. Similar to the proof of Lemma 2.3, we have

$$W_\alpha(x) = \sum_{n=1}^{[x]-1} \int_n^{n+1} t^\alpha(t-n-\frac{1}{2})dt + \int_{[x]}^x t^\alpha(t-[x]-\frac{1}{2})dt \tag{2.13}$$

$$\begin{aligned}
&= \left(\sum_{n=1}^{[x]-1}\int_n^{n+1} t^{\alpha+1}dt + \int_{[x]}^x t^{\alpha+1}dt\right) - \left(\sum_{n=1}^{[x]-1}(n+\frac{1}{2})\int_n^{n+1} t^\alpha dt + ([x]+\frac{1}{2})\int_{[x]}^x t^\alpha dt\right)\\
&= \int_1^x t^{\alpha+1}dt - \frac{1}{\alpha+1}\left(\sum_{n=1}^{[x]-1}(n+\frac{1}{2})\left((n+1)^{\alpha+1} - n^{\alpha+1}\right) + ([x]+\frac{1}{2})(x^{\alpha+1}-[x]^{\alpha+1})\right)\\
&= \frac{x^{\alpha+2}-1}{\alpha+2} - \frac{1}{\alpha+1}\left(\sum_{n=2}^{[x]}(n-\frac{1}{2})n^{\alpha+1} - \sum_{n=1}^{[x]-1}(n+\frac{1}{2})n^{\alpha+1}\right) - \frac{1}{\alpha+1}([x]+\frac{1}{2})(x^{\alpha+1}-[x]^{\alpha+1})\\
&= \frac{x^{\alpha+2}-1}{\alpha+2} - \frac{1}{\alpha+1}\left(-\sum_{n=2}^{[x]-1} n^{\alpha+1} + ([x]-\frac{1}{2})[x]^{\alpha+1} - \frac{3}{2}\right) - \frac{1}{\alpha+1}([x]+\frac{1}{2})(x^{\alpha+1}-[x]^{\alpha+1}).
\end{aligned}$$

We make some simplification, and obtain that (2.9) holds in this case. Next, if $1 \le x < 2$, then $[x] = 1$, it is easy to check that (2.9) also holds, and this completes the proof of (2.9).

Now we consider the case $\alpha = -2$. For $x \ge 2$, by the same method as above, we have

$$W_{-2}(x) = \log x - \psi(x)x^{-1} - \sum_{n\le x} n^{-1} + \frac{1}{2}. \tag{2.14}$$

If $1 \le x < 2$, then $W_{-2}(x) = \log x - \frac{3}{2} + \frac{3}{2}x^{-1}$, and (2.14) also holds in this case. This completes the proof of (2.10). Applying Euler-Maclaurin formula(see (2.20)) to the sum $\sum_{n\le x} n^{-1}$, we can get (2.11).

Finally, by applying integration by parts and (2.8), we have

$$\begin{aligned}
W_{-1}(x) = &\quad \psi_1(x)x^{-1} + \int_1^x t^{-2}\psi_1(t)dt \\
= &\quad \psi_1(x)x^{-1} + \psi_2(x)x^{-2} + 2\int_1^x t^{-3}\psi_2(t)dt \\
= &\quad 2\int_1^\infty t^{-3}\psi_2(t)dt + \psi_1(x)x^{-1} + \psi_2(x)x^{-2} - 2\int_x^\infty t^{-3}\psi_2(t)dt \\
= &\quad \int_1^\infty t^{-3}\psi_2(t)dt + \psi_1(x)x^{-1} + (-\frac{x}{12} + O(1))x^{-2} - 2\int_x^\infty \frac{-\frac{t}{12} + O(1)}{t^3}(t)dt.
\end{aligned}$$

This finishes the proof of Lemma 2.4.

When $\alpha = 0$, from (2.9) and some easy calculations, we get for $x \geq 1$ that

$$W_0(x) = \psi_1(x) = \frac{1}{2}(\psi^2(x) - \frac{1}{4}), \tag{2.15}$$

and

$$W_0(n) = \psi_1(n) = 0. \tag{2.16}$$

If $\alpha = 1$, we get for $x \geq 1$ that

$$\begin{aligned}
W_1(x) &= -\frac{x^3}{6} + \frac{1}{2}\frac{[x]([x]+1)(2[x]+1)}{6} + \frac{\psi(x)}{2}x^2 - \frac{1}{12} \\
&= -\frac{1}{24}x + \frac{1}{2}\psi^2(x)x - \frac{1}{6}\psi^3(x) + \frac{1}{24}\psi(x) - \frac{1}{12}.
\end{aligned} \tag{2.17}$$

When α is a non-negative integer, we may use the well-known Bernoulli polynomial to express $W_\alpha(x)$. Otherwise, we can use the following Lemma 2.5 to estimate it.

Lemma 2.5. Let $\alpha \neq -1, -2$, then for $x \geq 1$,

$$W_\alpha(x) = \frac{1}{\alpha+1}\left(\zeta(-1-\alpha) - \frac{\alpha}{2(2+\alpha)}\right) + (\psi_1(x) + \frac{1}{12})x^\alpha + O(x^{\alpha-1}) \tag{2.18}$$

Proof. From Euler-Maclaurin formula, for $s \neq 1$ and $x \geq 1$, we have

$$\sum_{n\leq x} n^{-s} = \zeta(s) + \frac{x^{1-s}}{1-s} - \psi(x)x^{-s} - s(\psi_1(x) + \frac{1}{12})x^{-s-1} + O(x^{-s-2}), \tag{2.19}$$

and

$$\sum_{n\leq x} n^{-1} = \log x + \gamma - \psi(x)x^{-1} - (\psi_1(x) + \frac{1}{12})x^{-2} + O(x^{-3}). \tag{2.20}$$

Now (2.18) is a immediate consequence of (2.9) and (2.19).

3 An asymptotic formula for the error term $\Delta(a,b;x)$

It is well-known that

$$\Delta(a,b;x) = -\sum_{n^{a+b}\le x} \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right) + \psi\left(\left(\frac{x}{n^a}\right)^{\frac{1}{b}}\right) + O(1). \tag{3.1}$$

However, the error function $O(1)$ in (3.1) is too large to prove our theorems. So we need a sharper form than (3.1). In this section we shall prove such a lemma.

Lemma 3.1. Let $(a,b)=1$ and $x\ge 1$, we define the error function $R(a,b;x)$ by

$$R(a,b;x) := \Delta(a,b;x) + \sum_{n^{a+b}\le x} \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right) + \psi\left(\left(\frac{x}{n^a}\right)^{\frac{1}{b}}\right). \tag{3.2}$$

Then one has

$$\begin{aligned} R(a,b;x) = {} & -\frac{(a+b)^2}{ab}\psi_1(x^{\frac{1}{a+b}}) + \frac{b(a+b)}{a^2}x^{\frac{1}{a}}\int_{x^{\frac{1}{a+b}}}^{\infty}\psi_1(t)t^{-2-\frac{b}{a}}dt \\ & + \frac{a(a+b)}{b^2}x^{\frac{1}{b}}\int_{x^{\frac{1}{a+b}}}^{\infty}\psi_1(t)t^{-2-\frac{a}{b}}dt. \end{aligned} \tag{3.3}$$

In particular, we have

$$R(a,b;x) = -\frac{(a+b)^2}{ab}\psi_1(x^{\frac{1}{a+b}}) - \frac{1}{12}\frac{a^2+b^2}{ab} + O\left(x^{-\frac{1}{a+b}}\right). \tag{3.4}$$

Furthermore, if $x^{\frac{1}{a+b}}$ is not an integer, then the derivative of $R(a,b;x)$ satisfies

$$\begin{aligned} R'(a,b;x) = {} & -\frac{a+b}{ab}\psi(x^{\frac{1}{a+b}})x^{\frac{1}{a+b}-1} - \frac{a^2+b^2}{ab}\psi_1(x^{\frac{1}{a+b}})x^{-1} \\ & -\frac{a+b}{12}x^{-1} + O\left(x^{-1-\frac{1}{a+b}}\right). \end{aligned} \tag{3.5}$$

Proof. By applying the Dirichlet hyperbola method, we easily obtain

$$\begin{aligned} D(a,b;x) &= \sum_{n\le x} d(a,b;n) = \sum_{m^a n^b\le x} 1 \\ &= \sum_{m^{a+b}\le x}\left[\left(\frac{x}{m^a}\right)^{\frac{1}{b}}\right] + \sum_{n^{a+b}\le x}\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right] - \left(\sum_{m^{a+b}\le x} 1\right)\left(\sum_{n^{a+b}\le x} 1\right) \\ &= -\sum_{m^{a+b}\le x}\psi\left(\left(\frac{x}{m^a}\right)^{\frac{1}{b}}\right) + \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right) \\ &\quad + x^{\frac{1}{b}}\sum_{m^{a+b}\le x}\frac{1}{m^{\frac{a}{b}}} + x^{\frac{1}{a}}\sum_{n^{a+b}\le x}\frac{1}{n^{\frac{b}{a}}} - \left[x^{\frac{1}{a+b}}\right]^2 - \left[x^{\frac{1}{a+b}}\right]. \end{aligned} \tag{3.6}$$

It is easy to see that the function $\psi_1(x)$ is a periodic function with period 1 and is therefore continuous. For real $x \geq 1$ and $s > 0$, by using Riemann -Stieltjes integration, and then integration by parts, we get that

$$\begin{aligned}\sum_{n\leq x} n^{-s} = & \int_{1-0}^{x} t^{-s} d[t] = [x]x^{-s} + s\int_{1}^{x} [t]t^{-s-1}dt \\ = & -\psi(x)x^{-s} + (x-\frac{1}{2})x^{-s} + s\int_{1}^{x} (t-\frac{1}{2})t^{-s-1}dt \\ & -s\left(\psi_1(x)x^{-s-1} + (s+1)\int_{1}^{x} t^{-s-2}\psi_1(t)dt\right).\end{aligned}$$

Hence, for $s > 0$

$$\sum_{n\leq x} n^{-s} = \begin{cases} \frac{2x^{1-s}-1-s}{2(1-s)} - \frac{\psi(x)}{x^s} - \frac{s\psi_1(x)}{x^{s+1}} - s(s+1)\int_1^x \frac{\psi_1(t)}{t^{s+2}}dt, & \text{if } s \neq 1 \\ \log x + \frac{1}{2} - \frac{\psi(x)}{x} - \frac{\psi_1(x)}{x^2} - 2\int_1^x \frac{\psi_1(t)}{t^3}dt, & \text{if } s = 1. \end{cases} \tag{3.7}$$

It is well-known that for $s > 0$ and $s \neq 1$

$$\zeta(s) = \frac{s+1}{2(s-1)} - s\int_{1}^{\infty} t^{-s-1}\psi(t)dt. \tag{3.8}$$

In addition, from (2.11) we have

$$\int_{1}^{\infty} t^{-2}\psi(t)dt = \frac{1}{2} - \gamma. \tag{3.9}$$

Integrating by parts again, we see that (3.8) and (3.9) are equivalent to

$$\int_{1}^{\infty} t^{-s-2}\psi_1(t)dt = \frac{1}{2s(s-1)} - \frac{\zeta(s)}{s(s+1)}, \text{ if } s > 0 \text{ and } s \neq 1, \tag{3.10}$$

and

$$\int_{1}^{\infty} t^{-3}\psi_1(t)dt = \frac{1}{2}\left(\frac{1}{2} - \gamma\right), \tag{3.11}$$

respectively. Inserting (3.10) and (3.11) into (3.7), we obtain for $s > 0$

$$\sum_{n\leq x} n^{-s} = \begin{cases} \frac{x^{1-s}}{1-s} + \zeta(s) - \frac{\psi(x)}{x^s} - \frac{s\psi_1(x)}{x^{s+1}} + s(s+1)\int_x^\infty \frac{\psi_1(t)}{t^{s+2}}dt, & \text{if } s \neq 1 \\ \log x + \gamma - \frac{\psi(x)}{x} - \frac{\psi_1(x)}{x^2} + 2\int_x^\infty \frac{\psi_1(t)}{t^3}dt, & \text{if } s = 1. \end{cases} \tag{3.12}$$

Now, taking $s = \frac{b}{a}$ and $s = \frac{a}{b}$ in (3.12) respectively, then combining (1.1), (2.7), (3.2), the following simple relations

$$\left[x^{\frac{1}{a+b}}\right] = x^{\frac{1}{a+b}} - \psi\left(x^{\frac{1}{a+b}}\right) - \frac{1}{2}$$

and

$$\left[x^{\frac{1}{a+b}}\right]^2 = x^{\frac{2}{a+b}} + \psi^2\left(x^{\frac{1}{a+b}}\right) + \frac{1}{4} - 2\psi\left(x^{\frac{1}{a+b}}\right)x^{\frac{1}{a+b}} - x^{\frac{1}{a+b}} + \psi\left(x^{\frac{1}{a+b}}\right),$$

we can get (3.3).

From formula (2.8) , it is easy to check that for $y \geq 1$

$$\psi_2(y) = \int_1^y \psi_1(t)dt = -\frac{y}{12} + O(1).$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_{x^{\frac{1}{a+b}}}^{\infty} \psi_1(t)t^{-\frac{b}{a}-2}dt \\
&= -\left(x^{\frac{1}{a+b}}\right)^{-\frac{b}{a}-2}\psi_2\left(x^{\frac{1}{a+b}}\right) + (2+\frac{b}{a})\int_{x^{\frac{1}{a+b}}}^{\infty}\psi_2(t)t^{-\frac{b}{a}-3}dt \\
&= -x^{-\frac{2a+b}{a(a+b)}}\left(-\frac{1}{12}x^{\frac{1}{a+b}} + O(1)\right) + (2+\frac{b}{a})\int_{x^{\frac{1}{a+b}}}^{\infty}\left(-\frac{t}{12} + O(1)\right)t^{-\frac{b}{a}-3}dt \\
&= -\frac{a}{12(a+b)}x^{-\frac{1}{a}} + O\left(x^{-\frac{1}{a}-\frac{1}{a+b}}\right).
\end{aligned}$$

Similarly, we also have

$$\int_{x^{\frac{1}{a+b}}}^{\infty} \psi_1(t)t^{-\frac{a}{b}-2}dt = -\frac{b}{12(a+b)}x^{-\frac{1}{b}} + O\left(x^{-\frac{1}{b}-\frac{1}{a+b}}\right).$$

Combining the above two estimates and (3.3) completes the proof of (3.4).

Finally, we suppose that $x^{\frac{1}{a+b}}$ is not an integer, thus $\psi_1\left(x^{\frac{1}{a+b}}\right)$ is differentiable. By differentiating the both sides of (3.3) with respect to x, and then applying the above two estimates, we have

$$\begin{aligned}
R'(a,b;x) = \quad & -\frac{(a+b)^2}{ab}\frac{1}{a+b}\psi(x^{\frac{1}{a+b}})x^{\frac{1}{a+b}-1} + \frac{b(a+b)}{a}x^{\frac{1}{a}-1}\int_{x^{\frac{1}{a+b}}}^{\infty}\psi_1(t)t^{-\frac{b}{a}-2}dt \\
& -\frac{b(a+b)}{a}x^{\frac{1}{a}}\psi_1(x^{\frac{1}{a+b}})x^{\frac{1}{a+b}(-\frac{b}{a}-2)}\frac{1}{a+b}x^{\frac{1}{a+b}-1} + \frac{a(a+b)}{b}x^{\frac{1}{b}-1}\int_{x^{\frac{1}{a+b}}}^{\infty}\psi_1(t)t^{-\frac{a}{b}-2}dt \\
& -\frac{a(a+b)}{b}x^{\frac{1}{b}}\psi_1(x^{\frac{1}{a+b}})x^{\frac{1}{a+b}(-\frac{a}{b}-2)}\frac{1}{a+b}x^{\frac{1}{a+b}-1} \\
= \quad & -\frac{a+b}{ab}\psi(x^{\frac{1}{a+b}})x^{\frac{1}{a+b}-1} - \frac{a^2+b^2}{ab}\psi_1(x^{\frac{1}{a+b}})x^{-1} - \frac{a+b}{12}x^{-1} + O\left(x^{-1-\frac{1}{a+b}}\right),
\end{aligned}$$

and this completes the proof of Lemma 3.1.

4 Integral formulas involving the ψ-function

In this section we shall evaluate integrals involving the ψ functions, which are important for our proof.

Lemma 4.1. Let n be an positive integer, α real, $x \geq 1$, and $n^{a+b} \leq x$, we define $I_{(a,b)}^{(\alpha)}(n,x) := \int_{n^{a+b}}^{x} t^{\alpha}\psi(t)\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right) dt$. Then

$$
I_{(a,b)}^{(\alpha)}(n,x) = \frac{1}{n^{\frac{b}{a}}}\left(W_{\alpha+\frac{1}{a}}(x) - W_{\alpha+\frac{1}{a}}(n^{a+b})\right) + \sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} W_{\alpha}(j^a n^b) + \tag{4.1}
$$

$$
(n-\frac{1}{2})W_{\alpha}(n^{a+b}) - \left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}} - \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\right) W_{\alpha}(x),
$$

where $W_{\alpha}(x)$ is defined by Lemma 2.4.

Proof. We first suppose $\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right] \geq n+1$. We divide the integral interval into the following subintervals, and obtain

$$
I_{(a,b)}^{(\alpha)}(n,x) = \sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1} \int_{j^a n^b}^{(j+1)^a n^b} t^{\alpha}\psi(t)\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right) dt \tag{4.2}
$$

$$+\int_{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b}^{x} t^{\alpha}\psi(t)\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right)dt$$

$$= \sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1} \int_{j^a n^b}^{(j+1)^a n^b} t^{\alpha}\psi(t)\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}-j-\frac{1}{2}\right)dt$$

$$+\int_{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b}^{x} t^{\alpha}\psi(t)\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}-\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-\frac{1}{2}\right)dt$$

$$= \frac{1}{n^{\frac{b}{a}}}\left(\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1} \int_{j^a n^b}^{(j+1)^a n^b} t^{\alpha+\frac{1}{a}}\psi(t)dt+\int_{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b}^{x} t^{\alpha+\frac{1}{a}}\psi(t)dt\right)$$

$$-\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1}(j+\frac{1}{2})\int_{j^a n^b}^{(j+1)^a n^b} t^{\alpha}\psi(t)dt-\left(\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]+\frac{1}{2}\right)\int_{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b}^{x} t^{\alpha}\psi(t)dt$$

$$= \frac{1}{n^{\frac{b}{a}}}\int_{n^{a+b}}^{x} t^{\alpha+\frac{1}{a}}\psi(t)dt-\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1}(j+\frac{1}{2})\left(W_{\alpha}((j+1)^a n^b)-W_{\alpha}(j^a n^b)\right)$$

$$-\left(\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]+\frac{1}{2}\right)\left(W_{\alpha}(x)-W_{\alpha}\left(\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b\right)\right).$$

Moreover, by Abel's summation formula, we get

$$\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1}(j+\frac{1}{2})\left(W_{\alpha}((j+1)^a n^b)-W_{\alpha}(j^a n^b)\right) \tag{4.3}$$

$$= \sum_{j=n+1}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]}(j-\frac{1}{2})W_{\alpha}(j^a n^b)-\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1}(j+\frac{1}{2})W_{\alpha}(j^a n^b)$$

$$= -\sum_{j=n+1}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-1}W_{\alpha}(j^a n^b)+\left(\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]-\frac{1}{2}\right)W_{\alpha}\left(\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]^a n^b\right)-(n+\frac{1}{2})W_{\alpha}(n^{a+b}).$$

Combining (4.2) and (4.3), we find that (4.1) holds in this case. If $\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]=n$, it is easy to check that (4.1) also holds, and this completes the proof of Lemma 4.1.

Lemma 4.2. Let $\alpha \neq -1, -2$, $n^{a+b} \leq x$, and $x \geq 1$. Then

$$(4.4) I^{(0)}_{(a,b)}(n,x) = \frac{1}{n^{\frac{b}{a}}}\left(W_{\frac{1}{a}}(x) - W_{\frac{1}{a}}(n^{a+b})\right) - \left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}} - \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\right)\psi_1(x).$$

If $\alpha \neq -\frac{1}{a}$, then

$$(4.5) \quad I^{(\alpha)}_{(a,b)}(n,x) = \frac{1}{12(1+a\alpha)}\left(\frac{x^{\alpha+\frac{1}{a}}}{n^{\frac{b}{a}}} - n^{(a+b)\alpha+1}\right) + \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\psi_1(x)x^{\alpha}$$

$$+O\left(\frac{x^{\alpha+\frac{1}{a}-1}}{n^{\frac{b}{a}}}\log x + x^{\alpha-\frac{1}{a}}n^{\frac{b}{a}} + n^{(a+b)\alpha-1} + n^{(a+b)(\alpha-1)+1}\log x\right),$$

and if $\alpha = -\frac{1}{a}$, then

$$(4.6) \quad I^{(-\frac{1}{a})}_{(a,b)}(n,x) = \frac{1}{12a}\frac{1}{n^{\frac{b}{a}}}(\log x - (a+b)\log n) + \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\psi_1(x)x^{-\frac{1}{a}}$$

$$+O\left(\frac{x^{-1}}{n^{\frac{b}{a}}}\log x + x^{-\frac{2}{a}}n^{\frac{b}{a}} + n^{-\frac{b}{a}-2} + n^{-(a+b)(\frac{1}{a}+1)+1}\log x\right).$$

Remark 2. In fact, the Lemma 3 of Furuya [5] is a special case of (4.4) with $a = 1$. In the present paper, when $\alpha = 0$, it is sufficient for us to apply a weaker estimate (4.5).

Proof. First, (4.4) is an immediate consequence of Lemma 4.1 and (2.16).

By Lemma 4.1, Lemma 2.5 and some simplification, we have

$$(4.7) \qquad I^{(\alpha)}_{(a,b)}(n,x) = \frac{1}{n^{\frac{b}{a}}}\left((\psi_1(x) + \frac{1}{12})x^{\alpha+\frac{1}{a}} - \frac{1}{12}n^{(a+b)(\alpha+\frac{1}{a})}\right)$$

$$
\begin{aligned}
& +\frac{1}{n^{\frac{b}{a}}} \times O\left((x+n^{a+b})^{(\alpha+\frac{1}{a}-1)}\right) \\
& +\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]}\left(\frac{1}{\alpha+1}\left(\zeta(-1-\alpha)-\frac{\alpha}{2(2+\alpha)}\right)+\frac{1}{12}(j^a n^b)^\alpha+O((j^a n^b)^{(\alpha-1)})\right) \\
& +(n-\frac{1}{2})\left(\frac{1}{\alpha+1}\left(\zeta(-1-\alpha)-\frac{\alpha}{2(2+\alpha)}\right)+\frac{1}{12}(n^{a+b})^\alpha+O((n^{a+b})^{(\alpha-1)})\right) \\
& -\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}-\psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\right)\left(\frac{1}{\alpha+1}\left(\zeta(-1-\alpha)-\frac{\alpha}{2(2+\alpha)}\right)+(\psi_1(x)+\frac{1}{12})x^\alpha+O(x^{(\alpha-1)})\right) \\
= \quad & -\frac{1}{24}n^{(a+b)\alpha}+\psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\left(\psi_1(x)+\frac{1}{12}\right))x^\alpha+\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]}\left(\frac{1}{12}(j^a n^b)^\alpha+O((j^a n^b)^{(\alpha-1)})\right) \\
& +O\left(\frac{x^{\alpha+\frac{1}{a}-1}}{n^{\frac{b}{a}}}+n^{(a+b)(\alpha-1)+1})\right).
\end{aligned}
$$

We write

$$
\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]}\left(\frac{1}{12}(j^a n^b)^\alpha+O((j^a n^b)^{(\alpha-1)})\right) \tag{4.8}
$$

$$
=\frac{1}{12}n^{b\alpha}\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} j^{a\alpha}+O\left(n^{b(\alpha-1)}\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} j^{a(\alpha-1)}\right).
$$

If $\alpha \neq -\frac{1}{a}$, by (2.19) we get

$$
\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} j^{a\alpha}=\sum_{j\leq\left(\frac{x}{n^b}\right)^{\frac{1}{a}}} j^{a\alpha}-\sum_{j\leq n} j^{a\alpha}+n^{a\alpha} \tag{4.9}
$$

$$
\begin{aligned}
&= \frac{\left(\frac{x}{n^b}\right)^{\frac{1}{a}(1+a\alpha)} - n^{1+a\alpha}}{1+a\alpha} - \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)^{a\alpha} \\
&\quad +\psi(n)n^{a\alpha} + n^{a\alpha} + O\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}(a\alpha-1)} + n^{a\alpha-1}\right) \\
&= \frac{\left(\frac{x}{n^b}\right)^{\alpha+\frac{1}{a}} - n^{1+a\alpha}}{1+a\alpha} - \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\left(\frac{x}{n^b}\right)^{\alpha} \\
&\quad +\frac{1}{2}n^{a\alpha} + O\left(\left(\frac{x}{n^b}\right)^{\alpha-\frac{1}{a}} + n^{a\alpha-1}\right).
\end{aligned}
$$

If $\alpha = -\frac{1}{a}$, by (2.20) we have

$$
\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} j^{a\alpha} = \sum_{j\le\left(\frac{x}{n^b}\right)^{\frac{1}{a}}} j^{-1} - \sum_{j\le n} j^{-1} + n^{-1} \tag{4.10}
$$

$$
= \frac{1}{a}\left(\log x - (a+b)\log n\right) - \psi\left(\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right)\left(\frac{x}{n^b}\right)^{-\frac{1}{a}} + \frac{1}{2n} + O(n^{-2}).
$$

From (4.9) and (4.10), we also have

$$
\sum_{j=n}^{\left[\left(\frac{x}{n^b}\right)^{\frac{1}{a}}\right]} j^{a(\alpha-1)} \ll \begin{cases} \left(\frac{x}{n^b}\right)^{\alpha+\frac{1}{a}-1} + n^{1+a(\alpha-1)}, & \text{if } \alpha \ne 1-\frac{1}{a} \\ \log\frac{x}{n^{a+b}} + \frac{1}{n}, & \text{if } \alpha = 1-\frac{1}{a} \end{cases} \tag{4.11}
$$

$$
\ll \left(\frac{x}{n^b}\right)^{\alpha+\frac{1}{a}-1}\log x + n^{1+a(\alpha-1)}\log x.
$$

Now (4.5) follows from (4.7),(4.8),(4.9) and (4.11), (4.6) follows from (4.7),(4.8),(4.10) and (4.11). This completes the proof of Lemma 4.2.

5 The proofs of Theorem 1 and Theorem 2

We first prove Theorem 1. We take $f(n) = d(a,b;n)$, $g(x) = \zeta(\frac{b}{a})x^{\frac{1}{a}} + \zeta(\frac{a}{b})x^{\frac{1}{b}}$ in Lemma 2.1. By Lemma 2.1 with $k=2$, we have

$$
\sum_{n\le x} \Delta^2(a,b;n) = \left(\frac{1}{2} - \psi(x)\right)\Delta^2(a,b;x) + \int_1^x \Delta^2(a,b;t)dt \tag{5.1}
$$

$$+2\int_1^x\left(\frac{1}{2}-\psi(t)\right)\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\Delta(a,b;t)dt$$
$$= \left(\frac{1}{2}-\psi(x)\right)\Delta^2(a,b;x)+\int_1^x\Delta^2(a,b;t)dt+T_1-2T_2,$$

where

$$T_1:=\int_1^x\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\Delta(a,b;t)dt, \tag{5.2}$$

and

$$T_2:=\int_1^x\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)\Delta(a,b;t)dt. \tag{5.3}$$

We treat T_1 first and shall show that

$$T_1=\frac{1}{4}\zeta(\frac{b}{a})x^{\frac{1}{a}}+\frac{1}{4}\zeta(\frac{a}{b})x^{\frac{1}{b}} \tag{5.4}$$
$$+\left(\frac{1}{a}\zeta(\frac{b}{a})x^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})x^{\frac{1}{b}-1}\right)G_{(a,b)}(x)+O\left(x^{\frac{1}{a}-\frac{3}{2(a+b)}}\right),$$

where the series $G_{(a,b)}(x)$ was defined in Theorem 1.

Integrating by parts, we have

$$T_1=\left(\frac{1}{a}\zeta(\frac{b}{a})x^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})x^{\frac{1}{b}-1}\right)\int_1^x\Delta(a,b;t)dt \tag{5.5}$$
$$-\int_1^x\left(\frac{1}{a}(\frac{1}{a}-1)\zeta(\frac{b}{a})t^{\frac{1}{a}-2}+\frac{1}{b}(\frac{1}{b}-1)\zeta(\frac{a}{b})t^{\frac{1}{b}-2}\right)\left(\int_1^t\Delta(a,b;u)du\right)dt.$$

We consider two cases.

Case(i). If $a=1$ and $b\geq 2$, by Lemma 2.2 with $q=1$, a simple splitting argument and the first derivative test(See (2.3) in Ivić[13])(Similar to the estimate of T_1^* below in this paper), we easily get

$$\begin{aligned}T_1= & \left(\zeta(b)+\frac{1}{b}\zeta(\frac{1}{b})x^{\frac{1}{b}-1}\right)\left(\frac{1}{4}x+G_{(a,b)}(x)+O\left(x^{1-\frac{3}{2(1+b)}}\right)\right)\\ & -\frac{1}{4b}(\frac{1}{b}-1)\zeta(\frac{a}{b})bx^{\frac{1}{b}}+O\left(x^{\frac{1}{b}-\frac{3}{2(1+b)}}\right)\\ = & \frac{1}{4}\zeta(b)x+\frac{1}{4}\zeta(\frac{1}{b})x^{\frac{1}{b}}+\left(\zeta(b)+\frac{1}{b}\zeta(\frac{1}{b})x^{\frac{1}{b}-1}\right)G_{(a,b)}(x)+O\left(x^{1-\frac{3}{2(1+b)}}\right).\end{aligned}$$

This proves that (5.4) holds in this case.

Case (ii). Suppose $a \geq 2$ and $b > a$. Similar to proof of the case (i), we also have

$$
\begin{aligned}
T_1 = \quad & \left(\frac{1}{a}\zeta(\frac{b}{a})x^{\frac{1}{a}-1} + \frac{1}{b}\zeta(\frac{a}{b})x^{\frac{1}{b}-1}\right)\left(\frac{1}{4}x + G_{(a,b)}(x) + O\left(x^{1-\frac{3}{2(a+b)}}\right)\right) \\
& -\frac{1}{4a}(\frac{1}{a}-1)\zeta(\frac{b}{a})ax^{\frac{1}{a}} - \frac{1}{4b}(\frac{1}{b}-1)\zeta(\frac{a}{b})bx^{\frac{1}{b}} + O\left(x^{\frac{1}{a}-\frac{3}{2(a+b)}}\right) \\
= \quad & \frac{1}{4}\zeta(\frac{b}{a})x^{\frac{1}{a}} + \frac{1}{4}\zeta(\frac{a}{b})x^{\frac{1}{b}} + \left(\frac{1}{a}\zeta(\frac{b}{a})x^{\frac{1}{a}-1} + \frac{1}{b}\zeta(\frac{a}{b})x^{\frac{1}{b}-1}\right)G_{(a,b)}(x) + O\left(x^{\frac{1}{a}-\frac{3}{2(a+b)}}\right).
\end{aligned}
$$

This completes the proof of (5.4).

Next we estimate T_2. From Lemma 3.1, we write

$$
\begin{aligned}
T_2 = \quad & -\int_1^x \left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1} + \frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)\left(\sum_{n^{a+b}\leq t}\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right) + \psi\left(\left(\frac{t}{n^a}\right)^{\frac{1}{b}}\right)\right)dt \\
& + \int_1^x \left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1} + \frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)R(a,b;t)dt
\end{aligned}
$$

$$
= T_{21} + T_{22}. \tag{5.6}
$$

To treat T_{22}, we divide the interval $[1,x]$ into two subsets I_1 and I_2, where $I_1 = [1,x]\backslash\bigcup_{j=1}^{[x^{\frac{1}{a+b}}]+1}[j^{a+b}-\frac{1}{10}, j^{a+b}+\frac{1}{10}]$, and $I_2 = [1,x]\bigcap\left(\bigcup_{j=1}^{[x^{\frac{1}{a+b}}]+1}[j^{a+b}-\frac{1}{10}, j^{a+b}+\frac{1}{10}]\right)$. For I_2, we have trivial estimate

$$
\begin{aligned}
& \int_{I_2}\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1} + \frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)R(a,b;t)dt \\
& \ll \int_{I_2} t^{\frac{1}{a}-1}dt \ll \sum_{j\leq x^{\frac{1}{a+b}}} j^{\frac{1}{a}-1} \ll x^{\frac{1}{a(a+b)}}.
\end{aligned}
$$

For I_1, by applying integration by parts and (3.5) in Lemma 3.1, we get

$$\int_{I_1}\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)R(a,b;t)dt$$

$$=\sum_{j\le x^{\frac{1}{a+b}}}\int_{[1,x]\bigcap[j^{a+b}+\frac{1}{10},(j+1)^{a+b}-\frac{1}{10}]}\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)R(a,b;t)dt$$

$$\ll\sum_{j\le x^{\frac{1}{a+b}}}\left(((j+1)^{a+b})^{(\frac{1}{a}-1)}+\int_{[1,x]\bigcap[j^{a+b}+\frac{1}{10},(j+1)^{a+b}-\frac{1}{10}]}|\psi_1(t)|\left(t^{\frac{1}{a}-2}+t^{\frac{1}{a}-1}|R'(a,b;t)|\right)dt\right)$$

$$\ll\sum_{j\le x^{\frac{1}{a+b}}}\left(((j+1)^{a+b})^{(\frac{1}{a}-1)}+\int_{j^{a+b}}^{(j+1)^{a+b}}t^{\frac{1}{a}-1}t^{\frac{1}{a+b}-1}dt\right)$$

$$\ll\sum_{j\le x^{\frac{1}{a+b}}}\left(((j+1)^{a+b})^{(\frac{1}{a}-1)}+\left|(j+1)^{(a+b)(\frac{1}{a}+\frac{1}{a+b}-1)}-j^{(a+b)(\frac{1}{a}+\frac{1}{a+b}-1)}\right|\right)$$

$$\ll\sum_{j\le x^{\frac{1}{a+b}}}\left(((j+1)^{a+b})^{(\frac{1}{a}-1)}+(j+1)^{(a+b)(\frac{1}{a}+\frac{1}{a+b}-1)-1}+j^{(a+b)(\frac{1}{a}+\frac{1}{a+b}-1)-1}\right)$$

$$\ll\sum_{j\le x^{\frac{1}{a+b}}}\left((j^{(a+b)})^{(\frac{1}{a}-1)}+(j+1)^{(a+b)(\frac{1}{a}-1)}\right)$$

$$\ll\begin{cases} x^{\frac{1}{a+b}}, & \text{if } a=1\\ 1, & \text{if } a\ge 2.\end{cases}$$

Combining the above two estimates, we get

$$T_{22}\ll x^{\frac{1}{a(a+b)}}. \tag{5.7}$$

To estimate T_{21}, we write

$$\begin{aligned}T_{21}&=-\int_1^x\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)\left(\sum_{n^{a+b}\le t}\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right)+\psi\left(\left(\frac{t}{n^a}\right)^{\frac{1}{b}}\right)\right)dt\\
&=-\sum_{n^{a+b}\le x}\int_{n^{a+b}}^x\left(\frac{1}{a}\zeta(\frac{b}{a})t^{\frac{1}{a}-1}+\frac{1}{b}\zeta(\frac{a}{b})t^{\frac{1}{b}-1}\right)\psi(t)\left(\psi\left(\left(\frac{t}{n^b}\right)^{\frac{1}{a}}\right)+\psi\left(\left(\frac{t}{n^a}\right)^{\frac{1}{b}}\right)\right)dt\\
&=-\sum_{n^{a+b}\le x}\frac{1}{a}\zeta(\frac{b}{a})\left(I_{(a,b)}^{(\frac{1}{a}-1)}(n,x)+I_{(b,a)}^{(\frac{1}{a}-1)}(n,x)\right)+\frac{1}{b}\zeta(\frac{a}{b})\left(I_{(a,b)}^{(\frac{1}{b}-1)}(n,x)+I_{(b,a)}^{(\frac{1}{b}-1)}(n,x)\right)\end{aligned}$$

$$:=-\frac{1}{a}\zeta(\frac{b}{a})(T_{211}+T_{212})-\frac{1}{b}\zeta(\frac{a}{b})(T_{213}+T_{214}), \tag{5.8}$$

and will show

$$(5.9)\ T_{21} = \begin{cases} -\frac{1}{12}\zeta(b)\left(\zeta(b)x + 2\zeta(\frac{1}{b})x^{\frac{1}{b}}\right) + O\left(x^{\frac{1}{1+b}}\right), & \text{if } a = 1, \text{ and } b \geq 2 \\ O(1+\log x), & \text{if } a \geq 2 \text{ and } b > a. \end{cases}$$

From (2.19) and (2.20), we easily obtain

$$(5.10) \qquad \sum_{n \leq x^{\frac{1}{a+b}}} n^{-\frac{b}{a}} = \begin{cases} \zeta(\frac{b}{a}) + O\left(x^{\frac{2}{a+b}-\frac{1}{a}}\right), & \text{if } b > a, \\ \frac{a}{a-b}x^{\frac{2}{a+b}-\frac{1}{a}} + O(1), & \text{if } b < a, \end{cases}$$

and

$$(5.11) \quad \sum_{n \leq x^{\frac{1}{a+b}}} n^{(a+b)\alpha+1} = \begin{cases} \frac{x^{\alpha+\frac{2}{a+b}}}{(a+b)\alpha+2} + O(1 + x^{\alpha+\frac{1}{a+b}}), & \text{if } \alpha > -\frac{2}{a+b}, \\ \frac{1}{a+b}\log x + \gamma + O(x^{-\frac{1}{a+b}}), & \text{if } \alpha = -\frac{2}{a+b}, \\ \zeta\left(-(a+b)\alpha - 1\right) + O(x^{\alpha+\frac{2}{a+b}}), & \text{if } \alpha < -\frac{2}{a+b}. \end{cases}$$

We will also use the following estimate

$$(5.12) \qquad \sum_{n \leq x} \frac{\log n}{n^{\frac{1}{2}}} = 2(\log x - 2)x^{\frac{1}{2}} + O(1),$$

which is an easy consequence of the Euler-Maclaurin formula.

Now we return to prove (5.9), and consider three cases.

Case (1). $a = 1$, $b = 2$.

By (4.5) in Lemma 4.2, (5.10) and (5.11), we get

$$\begin{aligned} T_{211} = & \sum_{n \leq x^{\frac{1}{3}}} I^{(0)}_{(1,2)}(n,x) \\ = & \sum_{n \leq x^{\frac{1}{3}}} \frac{1}{12}\left(\frac{x}{n^2} - n\right) + \psi\left(\frac{x}{n^2}\right)\psi_1(x) \\ & + O\left(\sum_{n \leq x^{\frac{1}{3}}} \frac{1}{n^2}\log x + x^{-1}n^2 + \frac{1}{n} + \frac{1}{n^2}\log x\right) \\ = & \frac{x}{12}\sum_{n \leq x^{\frac{1}{3}}} \frac{1}{n^2} - \frac{1}{12}\left(\frac{1}{2}x^{\frac{2}{3}} - \psi(x^{\frac{1}{3}})x^{\frac{1}{3}}\right) + \psi_1(x)\sum_{n \leq x^{\frac{1}{3}}} \psi\left(\frac{x}{n^2}\right) + O(\log x). \end{aligned}$$

(Here we use $\sum_{n \leq x^{\frac{1}{3}}} n = \frac{1}{2}x^{\frac{2}{3}} - \psi(x^{\frac{1}{3}})x^{\frac{1}{3}} + O(1)$). By (2.19) again, we obtain

$$(5.13) \qquad T_{211} = \frac{1}{12}\zeta(2)x - \frac{1}{8}x^{\frac{2}{3}} + O(x^{\frac{1}{3}}).$$

By (4.5) of Lemma 4.2, (5.10), (5.11) and (2.19), we have

$$
\begin{aligned}
T_{212} = &\quad \sum_{n\le x^{\frac{1}{3}}} I^{(0)}_{(2,1)}(n,x) \\
= &\quad \sum_{n\le x^{\frac{1}{3}}} \frac{1}{12}\left(\frac{x^{\frac{1}{2}}}{n^{\frac{1}{2}}} - n\right) + \psi\left(\left(\frac{x}{n}\right)^{\frac{1}{2}}\right)\psi_1(x) \\
&\quad +O\left(\sum_{n\le x^{\frac{1}{3}}} \frac{x^{-\frac{1}{2}}}{n^{\frac{1}{2}}}(1+\log x) + x^{-\frac{1}{2}}n^{\frac{1}{2}} + \frac{1}{n} + \frac{1}{n^2}\log x\right) \\
= &\quad \frac{1}{12}\sum_{n\le x^{\frac{1}{3}}} \frac{x^{\frac{1}{2}}}{n^{\frac{1}{2}}} - \frac{1}{12}\left(\frac{1}{2}x^{\frac{2}{3}} - \psi(x^{\frac{1}{3}})x^{\frac{1}{3}}\right) + \psi_1(x)\sum_{n\le x^{\frac{1}{3}}}\psi\left(\left(\frac{x}{n}\right)^{\frac{1}{2}}\right) + O(\log x) \\
= &\quad \frac{1}{8}x^{\frac{2}{3}} + \frac{1}{12}\zeta(\frac{1}{2})x^{\frac{1}{2}} + O(x^{\frac{1}{3}}).
\end{aligned}
$$

In the same way, by (4.5) in Lemma 4.2, (5.10), (5.11) and (2.19), we have

$$
\begin{aligned}
T_{213} = &\quad \sum_{n\le x^{\frac{1}{3}}} I^{(-\frac{1}{2})}_{(1,2)}(n,x) \\
= &\quad \frac{1}{6}\sum_{n\le x^{\frac{1}{3}}}\left(\frac{x^{\frac{1}{2}}}{n^2} - n^{-\frac{1}{2}}\right) + \psi_1(x)x^{-\frac{1}{2}}\sum_{n\le x^{\frac{1}{3}}}\psi\left(\frac{x}{n^2}\right) + O(\log x) \\
= &\quad \frac{1}{6}\zeta(2)x^{\frac{1}{2}} + O(x^{\frac{1}{3}}).
\end{aligned}
$$

By (4.6) in Lemma 4.2, (5.10), (5.11) and (5.12), we have

$$
\begin{aligned}
T_{214} = &\quad \sum_{n\le x^{\frac{1}{3}}} I^{(-\frac{1}{2})}_{(2,1)}(n,x) \\
= &\quad \frac{1}{24}\sum_{n\le x^{\frac{1}{3}}}\frac{\log x - 3\log n}{n^{\frac{1}{2}}} + \psi_1(x)x^{-\frac{1}{2}}\sum_{n\le x^{\frac{1}{3}}}\psi\left(\left(\frac{x}{n}\right)^{\frac{1}{2}}\right) + O(\log x) \\
= &\quad \frac{1}{2}x^{\frac{1}{6}} + +O(\log x).
\end{aligned}
$$

Combining the above four estimates , we see that (5.9) holds in this case.

Case (2). $a = 1$, $b \ge 3$.

By (4.5) in Lemma 4.2, (5.10), (5.11) and (2.19), we can get the following four

estimates:

$$
\begin{aligned}
T_{211} = & \sum_{n\le x^{\frac{1}{1+b}}} I^{(0)}_{(1,b)}(n,x)\\
= & \frac{1}{12}\sum_{n\le x^{\frac{1}{1+b}}}\left(\frac{x}{n^b}-n\right)+\psi_1(x)\sum_{n\le x^{\frac{1}{1+b}}}\psi\left(\frac{x}{n^b}\right)+O(\log x)\\
= & \frac{1}{12}\zeta(b)x-\frac{1}{12}(\frac{1}{b-1}+\frac{1}{2})x^{\frac{2}{1+b}}+O(x^{\frac{1}{1+b}}),
\end{aligned}
$$

$$
\begin{aligned}
T_{212} = & \sum_{n\le x^{\frac{1}{1+b}}} I^{(0)}_{(b,1)}(n,x)\\
= & \frac{1}{12}\sum_{n\le x^{\frac{1}{1+b}}}\left(\frac{x^{\frac{1}{b}}}{n^{\frac{1}{b}}}-n\right)+\psi_1(x)\sum_{n\le x^{\frac{1}{1+b}}}\psi\left(\left(\frac{x}{n}\right)^{\frac{1}{b}}\right)+O(\log x)\\
= & \frac{1}{12}\left(\frac{1}{b-1}+\frac{1}{2}\right)x^{\frac{2}{b+1}}+\frac{1}{12}\zeta(\frac{1}{b})x^{\frac{1}{b}}+O(x^{\frac{1}{b+1}}),
\end{aligned}
$$

$$
\begin{aligned}
T_{213} = & \sum_{n\le x^{\frac{1}{1+b}}} I^{(-1+\frac{1}{b})}_{(1,b)}(n,x)\\
= & \frac{b}{12}\sum_{n\le x^{\frac{1}{1+b}}}\left(\frac{x^{\frac{1}{b}}}{n^b}-n^{-b+1+\frac{1}{b}}\right)+\psi_1(x)x^{-1+\frac{1}{b}}\sum_{n\le x^{\frac{1}{1+b}}}\psi\left(\frac{x}{n^b}\right)+O(\log x)\\
= & \frac{b}{12}\zeta(b)x^{\frac{1}{b}}+O(\log x),
\end{aligned}
$$

and

$$
\begin{aligned}
T_{214} = & \sum_{n\le x^{\frac{1}{1+b}}} I^{(-1+\frac{1}{b})}_{(b,1)}(n,x)\\
= & \frac{1}{12(2-b)}\sum_{n\le x^{\frac{1}{1+b}}}\left(\frac{x^{-1+\frac{2}{b}}}{n^{\frac{1}{b}}}-n^{-b+1+\frac{1}{b}}\right)+\psi_1(x)x^{-1+\frac{1}{b}}\sum_{n\le x^{\frac{1}{1+b}}}\psi\left(\left(\frac{x}{n}\right)^{\frac{1}{b}}\right)+O(\log x)\\
= & O(\log x).
\end{aligned}
$$

Hence, (5.9) also holds in this case.

Case (3). $a\ge 2$, $b\ge 3$.

Similar to the proof of Case (2), we can prove

$$T_{211}, T_{212}, T_{213}, T_{214} \ll \log x. \tag{5.14}$$

We omit the details. This completes the the proof of (5.9).

Note that if $a = 1, b \geq 2$, then $\frac{1}{1+b} < 1 - \frac{3}{2(1+b)}$; if $a \geq 2, b > a$, then $\frac{1}{a(a+b)} < \frac{1}{a} - \frac{3}{2(a+b)}$. Now, collecting (5.1), (5.5)-(5.9) completes the proof of Theorem 1.

Finally, we shall give a short proof of Theorem 2, since the details are similar to and simpler than that of Theorem 1.

We take $f(n) = d(1,1;n)$, $g(x) = (\log x + 2\gamma - 1)x$. By Lemma 2.1 with $k = 2$, we have

$$\sum_{n \leq x} \Delta^2(1,1;n) = \left(\frac{1}{2} - \psi(x)\right) \Delta^2(1,1;x) + \int_1^x \Delta^2(1,1;t)dt + T_1^* + T_2^*, \tag{5.15}$$

where

$$T_1^* := \int_1^x (\log t + 2\gamma)\Delta(1,1;t)dt, \tag{5.16}$$

and

$$T_2^* := -2\int_1^x (\log t + 2\gamma)\psi(t)\Delta(1,1;t)dt. \tag{5.17}$$

Similar to the estimate of T_2, we may get that(Also see Furuya[5],page 17-18)

$$T_2^* := \frac{1}{6}x\log^2 x + \frac{1}{3}(2\gamma - 1)x\log x + \frac{1}{3}(2\gamma^2 - 2\gamma + 1)x + O(x^{\frac{1}{2}}\log x). \tag{5.18}$$

Now we estimate T_1^* as T_1 . By Lemma 2.2 with $q = 1$, and then integrating by parts, we find that

$$\begin{aligned}
T_1^* = \quad & \int_1^x (\log t + 2\gamma)\Delta(1,1;t)dt \\
= \quad & (\log x + 2\gamma)\int_1^x \Delta(1,1;t)dt - \int_1^x t^{-1}\left(\int_1^t \Delta(1,1;u)du\right)dt \\
= \quad & (\log x + 2\gamma)\left(\frac{1}{4}x + G_{(1,1)}(x) + O(x^{\frac{1}{4}})\right) - \int_1^x t^{-1}\left(\frac{t}{4} + G_{(1,1)}(t) + O(t^{\frac{1}{4}})\right)dt \\
= \quad & \frac{1}{4}\left(\log x + 2\gamma - 1\right)x + (\log x + 2\gamma)G_{(1,1)}(x) - \int_1^x t^{-1}G_{(1,1)}(t)dt + O(x^{\frac{1}{4}}\log x),
\end{aligned}$$

where

$$G_{(1,1)}(x) := \frac{1}{2\sqrt{2}\pi^2}x^{\frac{3}{4}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{5}{4}}}\sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right).$$

Since the series $G_{(1,1)}(x)$ is absolute convergent, one may integrate term by term, and obtain

$$\int_1^x t^{-1} G_{(1,1)}(t)dt = \frac{1}{2\sqrt{2}\pi^2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{5}{4}}} \int_1^x t^{-\frac{1}{4}} \sin\left(4\pi\sqrt{nt} - \frac{\pi}{4}\right) dt.$$

By a simple splitting argument and the first derivative test(see (2.3), Ivić[13]), we easily get

$$\int_1^x t^{-1} G_{(1,1)}(t)dt \ll \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{5}{4}}} \frac{x^{\frac{1}{4}}}{\sqrt{n}} \ll x^{\frac{1}{4}}. \tag{5.19}$$

Hence,

$$T_1^* = \frac{1}{4}\left(\log x + 2\gamma - 1\right) x + (\log x + 2\gamma) G_{(1,1)}(x) + O(x^{\frac{1}{4}}). \tag{5.20}$$

Combining (5.15), (5.18) and (5.20) completes the proof of Theorem 2.

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